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LETTER TO THE EDITOR

Computing the topological pressure for intermittent maps

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Abstract. The topological pressure is obtained as the leading zero of a dynamical zeta function. We consider the problem of computing this zero when it is close to a singularity. In particular we study a family of intermittent maps, which we argue exhibit a branch point singularity in its zeta functions. The convergence of the cycle expansion close to this point is extremely slow. To deal with this problem we consider a resummation of the cycle expansion. The idea is quite similar to that of Padé approximants, but the ansatz is a generalized series expansion around the branch point rather than a rational function. The improvement on convergence of the leading zero is considerable. We also briefly discuss the relation between correlation decay and the nature of the branch point.

1. Introduction

The topological pressure [1, 2] is related to the leading zero $z_0(\beta)$ of the zeta function

$$1/\zeta_\beta(z) = \prod_p \left(1 - \frac{z^{n_p}}{|\Lambda_p|^\beta}\right) \tag{1}$$

as a function of the parameter β. The product in (1) runs over all primitive periodic orbits p, having period n_p and stability $\Lambda_p = \mathrm{d}f^{n_p}/\mathrm{d}x|_{x=x_p}$ with x_p being any point along p. The topological pressure $\mathcal{P}(\beta)$ is given by

$$\mathcal{P}(\beta) = -\log z_0(\beta). \tag{2}$$

Various properties known from the thermodynamic formalism for chaotic system, such as Renyi entropies, Lyapunov exponents, are directly related to the topological pressure [2]. For instance the Lyapunov exponent is given by

$$\lambda = -\left.\frac{\mathrm{d}\mathcal{P}(\beta)}{\mathrm{d}\beta}\right|_{\beta=1} \tag{3}$$

and the topological entropy by

$$h = \mathcal{P}(0). \tag{4}$$

For open systems also the escape rate and (multi)fractal dimensions of the repellor are available via the the topological pressure.

By expanding the product (1) one obtains a formal power series representation of the zeta function, usually referred to as a cycle expansion [3]. The leading zero $z_0(\beta)$ can in principle be computed from this series, provided it lies inside the leading singularity, which is normally the case if $\beta < 1$. For Axiom A systems, that is, hyperbolic systems with a finite Markov partition, this procedure is computationally very efficient [3], in particular if one substitutes the Fredholm determinant for the zeta function. However, convergence is seriously slowed down if the leading zero is close to a singularity, or if zeros accumulate or cluster close by. We will now discuss how this problem may be dealt with for intermittent systems.

We will consider the intermittent map $x \mapsto f(x)$ on the unit interval, with

$$f(x) = \begin{cases} f_0(x) = x + 2^s x^{1+s} & 0 \leqslant x < 1/2 \\ f_1(x) = 2x - 1 & 1/2 \leqslant x \leqslant 1 \end{cases} \tag{5}$$

where $s > 0$. For $s = 0$ the map would just be the binary shift map and the leading zero given by $z_0(\beta) = 2^{\beta-1}$ and the topological pressure a linear function in β.

But for $s > 0$ the map is intermittent; the fixed point $x = 0$ is neutrally stable: $f'(0) = 1$. The map admits a binary coding, we associate the letter 0 with the left leg, and 1 with the right leg. The neutral fixed point now corresponds to the periodic orbit $\bar{0}$.

There are several methods for accelerating convergence of power series, for instance the method of Padé approximants is widely used. However, if the nature of the disturbing singularity is known one can tailor a more optimal resummation scheme. Therefore, we start in section 2 by studying the nature of the leading singularity which is found to be a branch point at $z = 1$. In section 3 we resum the formal power series (around $z = 0$) into a generalized power series around the branch point in a manner quite similar to that of Padé approximants. The leading zero is then computed from a truncation of the resummed series. Although the basic idea is rather simple, the resulting method is surprisingly efficient.

2. The nature of the leading singularity

The system under consideration has a complete binary symbolic dynamics, that is, all allowed orbits can be built from the alphabet $\{0, 1\}$. However, for our intermittent systems it is more natural to use the alphabet $\{0^n 1\}$, with $n \geqslant 0$. With this alphabet we can build all the periodic orbits except for the neutrally stable one $\bar{0}$. We will, therefore, extract the factor $(1 - z/\Lambda_0^\beta) = (1 - z)$ from $1/\zeta_\beta(z)$

$$1/\zeta_\beta(z) = (1 - z)/\tilde{\zeta}_\beta(z) \tag{6}$$

and consider the zeta function

$$1/\tilde{\zeta}_\beta(z) = \prod_{p \neq \bar{0}} \left(1 - \frac{z^{n_p}}{|\Lambda_p|^\beta}\right). \tag{7}$$

If we did the same thing for a hyperbolic system we would find that $1/\tilde{\zeta}_\beta(z)$ exhibited a pole that would be cancelled by $(1 - z/\Lambda_0^\beta)$, so in that case the original alphabet would be preferable. However, in our intermittent system such a cancellation will not occur.

In the following we will suppress the index β and instead index the zeta function by that map it refers to. We will consider the series expansion of the functions $1/\tilde{\zeta}_f(z)$

$$1/\tilde{\zeta}_f(z) = 1 - \sum_{n=1}^{\infty} a_n z^n. \tag{8}$$

The nature of the leading singularity will be reflected in the asymptotic behaviour of the coefficients $\{a_n\}$ of this power series. To get an idea what this asymptotic behaviour may be, we consider the zeta function of a closely related map $\hat{f}$, being a piecewise linear approximation of f. We define $\hat{f}$ as a continuous function, coinciding with f on a sequence of points $\hat{f}(c_n) = f(c_n)$ where the c_n's are the inverse images of the critical point

$$c_0 = 1 \tag{9}$$

$$c_{n+1} = f_0^{-1}(c_n) \tag{10}$$

and linear in the intervals $[c_{n+1}, c_n]$.

The dependence of the stabilities $\Lambda_{p=\overline{0^n1}}$ on n can be estimated by replacing the difference equation $x_{n+1} = x_n + 2^s x_n^{1+s}$ by the differential equation [5]

$$\frac{\mathrm{d}x_n}{\mathrm{d}n} = 2^s x_n^{1+s} \tag{11}$$

having solution

$$x_n = [x_0^{-s} - s2^s n]^{-1/s}. \tag{12}$$

Since now $x_n \sim c_0 \sim 1$ and $x_0 \sim c_n$ we are led to the following asymptotic behaviour

$$c_n \sim n^{-1/s}. \tag{13}$$

This scaling law can be verified rigorously [6].

The construction of $\hat{f}$ gives it a very simple cycle expansion

$$1/\tilde{\zeta}_{\hat{f}}(z) = 1 - \sum_{n=0}^{\infty} \frac{z^{n+1}}{|\Lambda_{\overline{0^n1}}|^{\beta}} \equiv \sum_n \hat{a}_n z^n. \tag{14}$$

The stabilities $\Lambda_{\overline{0^n1}}$ are simply related to the c_n's

$$\Lambda_{\overline{0^n1}} = \frac{1}{c_n - c_{n+1}} \tag{15}$$

with the asymptotic behaviour $\Lambda_{\overline{0^n1}} \sim n^{(s+1)/s}$. The asymptotic behaviour of the coefficients is thus

$$\hat{a}_n \sim n^{-\beta(s+1)/s} \tag{16}$$

and it is natural to expect that the same holds for the sequence $\{a_n\}$ in equation (8). If this is indeed the case, then $1/\tilde{\zeta}_f(z)$ will contain a singularity of the type

$$\begin{array}{ll} (1-z)^{\alpha} & \alpha \notin N \\ (1-z)^{\alpha}\log(1-z) & \alpha \in N \end{array} \tag{17}$$

with

$$\alpha(\beta, s) = \frac{\beta(s+1)}{s} - 1 \tag{18}$$

as can be realized through the Tauberian theorems for power series.

In [6] there are also stronger arguments that $z = 1$ is a branch point of this type but it is still not strictly proven. In [7] it is proven that the zeta function $1/\zeta_f(z)$ is holomorphic in a region enclosing the unit circle except for the line $[1, \infty]$ where we expect a cut. The results in [7] apply to a more general case without a complete symbolic coding.

To check the accuracy of the piecewise linear approximation we want to numerically determine the coefficients $\{a_n\}$ by simply expanding the product (7) using all periodic orbits up to period 20. This set contains $\sim 10^5$ periodic orbits and can be computed relatively fast. Periodic orbits are determined by a Newton–Raphson procedure. To this end we look for fixed points of some iterate of the inverted map, choosing branch according to the symbol code.

The coefficients for two values of s are plotted in figure 1, together with the piecewise linear approximation, which we know achieve the slope (16). The good agreement with the piecewise linear approximation support our claim that the singularity is of the suggested type.

For the binary shift map ($s = 0$) we would have $a_n = 1/2^n$. For a slightly higher value of the parameter $s = 0.1$ (figure 1(a)) the coefficients conform with this behaviour for small n, but eventually they are approaching the expected powerlaw.

We will also consider the Fredholm determinant

$$\tilde{F}(z) = \prod_{m=0}^{\infty} \prod_{p \neq \bar{0}} \left(1 - \frac{z^{n_p}}{|\Lambda_p|\Lambda_p^m}\right). \tag{19}$$

In the previous case, it was not essential to factor out the neutral periodic orbit, but when considering the Fredholm determinant it is essential, the extra factor $(1-z)^\infty$ would of course be devastating for our investigations.

In figure 1 we also plot the coefficient $\{b_i\}$ of the expansion of the Fredholm determinant

$$\tilde{F}(z) = 1 - \sum_{i=1}^{\infty} b_i z^i \tag{20}$$

together with the piecewise linear approximation and the good agreement leads us to assume that the Fredholm determinant share the same leading singularity as the dynamical zeta function.

3. Computing the topological pressure

Consider again the problem of computing the leading zero of

$$1/\tilde{\zeta}(z, \beta) = \prod_{p \neq \bar{0}} \left(1 - \frac{z^{n_p}}{|\Lambda_p|^\beta}\right). \tag{21}$$

If β is close to, but less than unity, this leading zero will be close to $z = 1$, and it will be extremely inefficient to compute it from a truncated power series in z. It is natural to try to expand $1/\tilde{\zeta}(z, \beta)$ in a generalized power series around $z = 1$. If the leading singularity is of the form $(1-z)^\alpha$ the simplest possible expansion would be

$$1/\tilde{\zeta}(z, \beta) = \sum_{i=0}^{\infty} c_i (1-z)^i + (1-z)^\alpha \sum_{i=0}^{\infty} d_i (1-z)^i. \tag{22}$$

According to our previous findings we expect that $\alpha = \beta(s+1)/s - 1$. Suppose now that we replace these infinite sums by finite sums of increasing degrees, n_c and n_d, and require that

$$\sum_{i=0}^{n_c} c_i (1-z)^i + (1-z)^\alpha \sum_{i=0}^{n_d} d_i (1-z)^i = 1/\tilde{\zeta}(z, \beta) + \mathrm{O}(z^{n+1}). \tag{23}$$

If $n_c + n_d + 2 = n + 1$ we just get a linear system of equation to solve in order to determine the coefficients c_i and d_i from those of the series expansion around $z = 0$. It is also natural to require that $|n_d + \alpha - n_c| < 1$. So far we have assumed that α is a noninteger. The case with integer α can be worked out in close analogy.

To test the idea we choose (arbitrarily) the parameters $s = 0.7$ and $\beta = 0.9$. In figure 2 we plot the leading zero against truncation length n, determined from expansion (23), and the expansion around $z = 0$. The improvement is obvious. However, we do not claim that

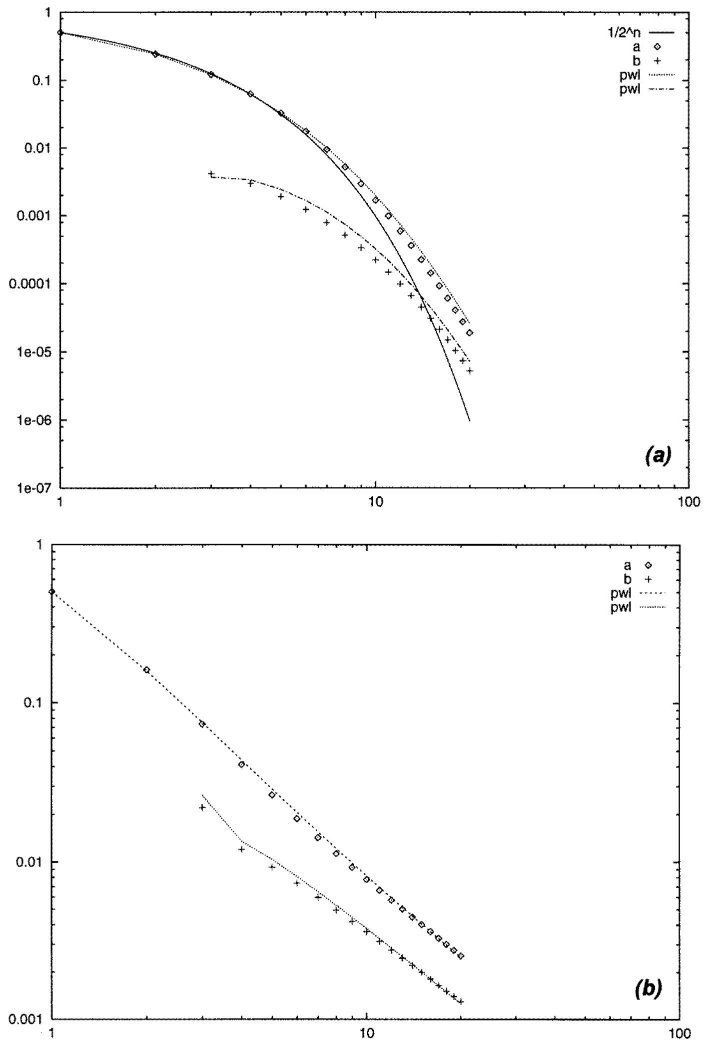

Figure 1. Expansion coefficients a_n of the dynamical zeta function $1/\tilde{\zeta}(z)(\beta = 1)$ and b_n of the Fredholm determinant $\tilde{F}(z)$ plotted against n for the parameter values: $s = 0.1$ (a), and $s = 2$ (b), together with the corresponding piecewise linear approximation (pwl). In (a) the corresponding sequence a_n for the case $s = 0$ is drawn for comparison.

the simple expansion (22) is entirely able to capture the singularity at $z = 1$. However, a more detailed understanding of the singularity structure could immediately be built in to the ansatz and convergence should be further accelerated.

In figure 3 we plot the resulting topological pressure against β. The dependence on s is more clearly seen if we plot the generalized Lyapunov exponents [2], related to the topological pressure according to

$$\lambda(1-\beta) = \frac{\mathcal{P}(\beta)}{1-\beta}. \tag{24}$$

The (ordinary) Lyapunov exponent is given by $\lambda = \lambda(0^+)$. The generalized Lyapunov exponents are plotted in figure 4.

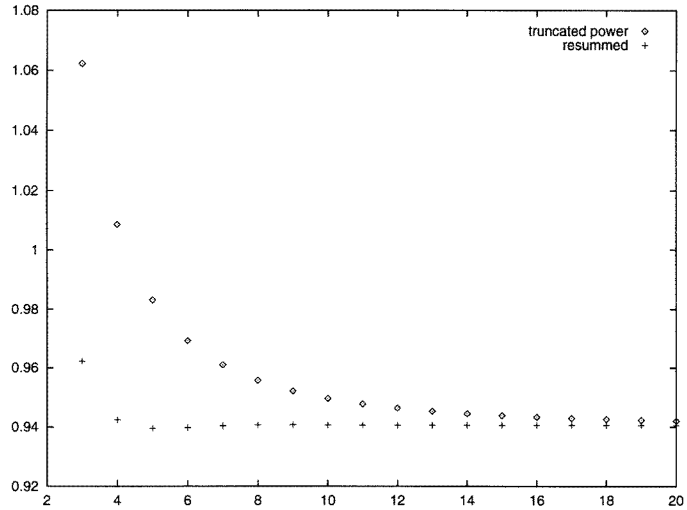

Figure 2. Leading zero for $s = 0.7$ and $\beta = 0.9$ against truncation length determined from the generalized series expansion around $z = 1$ and the power series around $z = 0$.

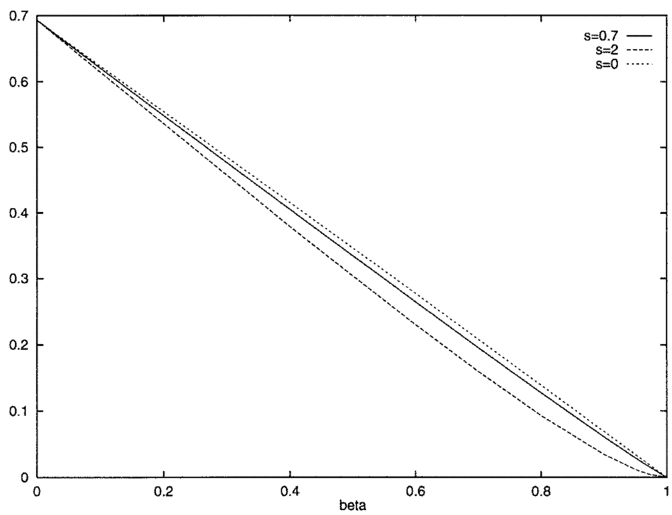

Figure 3. Topological pressure $\mathcal{P}(\beta)$.

If $\beta > 1$ the topological pressure will be governed by the branch point itself which stays fixed at $z = 1$ so $\mathcal{P}(\beta) = 0$ when $\beta > 1$. The nonanalyticity of $\mathcal{P}(\beta)$ at $\beta = 1$, related to the collision of the leading zero with the branch cut, is referred to as a phase transition [8].

4. A few words about correlation decay

The decay rate of correlations is a more intricate problem from a mathematical point of view than the computation of chaotic averages. It is known that the spectra of zeta functions ($\beta = 1$) and Fredholm determinants (19) are, at least in some cases, related to the (typical) decay of correlations. For an exponentially mixing system the mixing rate is given by the size of the *gap* between the leading and next-to-leading zeros of $F(z)$ [9]. We believe that a

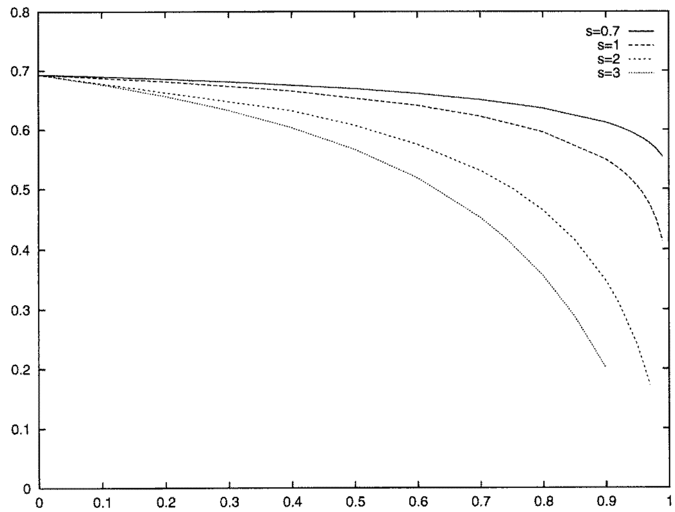

Figure 4. Generalized Lyapunov exponent $\lambda(1-\beta)$.

close coupling between typical correlation decay and the analytic structure of the dynamical zeta function or Fredholm determinant around $z = 1$ is possible even for systems without a gap. Let us take the (formal) trace of the transfer operator

$$\operatorname{tr}\mathcal{L}^n = \int_\epsilon^1 \delta(x - f^n(x))\,\mathrm{d}x = \sum_{p\neq\bar{0}} n_p \sum_{r=1}^{\infty} \frac{\delta_{n,rn_p}}{|\det(1-\Lambda_p^r)|} = \frac{1}{2\pi\mathrm{i}} \int_{|z|<1} z^{-n} \frac{\tilde{F}'(z)}{\tilde{F}(z)}\,\mathrm{d}z \qquad (25)$$

where ϵ is a small number. The trace is formal because it makes no explicit reference to eigenvalues of an operator, it is just the trace over the integral kernel of the operator. We claim that this trace serves as an archetype correlation function. If there is a gap, residue calculus tells us that the trace will approach unity exponentially fast and the rate is provided by the size of the gap. But could this really work for the intermittent case, where the leading zero is connected by a branch cut, running along the line $\mathrm{Im}\,(z) = 0\mathrm{Re}\,(z) > 1$? Let us assume that $\tilde{F}(z)$ is holomorphic and zero-free, except along the cut, in the disk $1 < |z| < C$ where $C > 1$ [7]. The value of (25) for large n will be governed by the vicinity of $z = 1$, which we assume is, to leading order, described (17). The result (25) can then be evaluated asymptotically

$$\operatorname{tr}\mathcal{L}^n \sim \begin{cases} 1 + C/n^{1/s-1} & 0 < s < 1 \\ 1 + C/\log n & s = 1 \\ 1/s & s > 1. \end{cases} \qquad (26)$$

For $0 < s < 1$ this suggests that typical correlations should decay as $\sim 1/n^{1/s-1}$ which indeed agrees with the rigorous results [6]. The failure of the trace to approach unity for $s > 1$ reflect the fact that the invariant density is not normalizable anymore. The relation between the topological pressure and thermodynamic properties mentioned in the introduction is then rather obscure [2]. Some anomalous properties for the case $s > 0$ are discussed in [4, 5].

There are indications that the identification between the behaviour of the formal trace and the typical correlation functions is possible also for the Sinai billiard which seems to exhibit the decay law $C(t) \sim 1/t$ [10, 11].

5. Concluding remarks

The zeta function is a fundamental object in chaos theory. In this paper we have tried to argue that even singularities in zeta functions carry interesting information. Moreover, the damaging effect they have on convergence can be overcome.

We have focused on the problem of intermittency and studied a system with a complete symbolic coding. We expect that the presented method should work even if the coding was not a complete one. But the determination of the leading zero would be disturbed by nonleading zeros and even singularities [12] beyond the unit circle.

In more general systems, in more dimensions and/or with continuous time, one can usually not even define a generating partition. Periodic orbits are difficult to find and authors have chosen alternative methods to access the topological pressure, via generalized Lyapunov exponents [13] or via correlation functions [14].

However, for intermittent systems, when the leading zero is close to the branch point the approach of [14] is not so successful. Phase transitions are difficult to extract from correlation data. In [10, 15, 16] we have suggested a way to overcome this difficulty. We have developed a probabalistic approximation of the zeta function, yielding a good description of the exact zeta function in the neighbourhood of the branchpoint. To that end we abandoned the periodic orbits entirely. We could then compute a few terms in the generalized power series accurately. The technique discussed in the present paper is more powerful. However, the expanded zeta functions for the continuous time systems considered in [10, 15, 16] are actually Dirichlet series and the technique of this paper is not directly applicable.

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